Revision on electromagnetic waves

FORCE AND POTENTIAL ENERGY

One dimensional case (1D)

$$E_p(x) = -\int_{\infty}^{x} F_x \, dx$$

Generalization to 3D

$$E_p(\vec{\mathbf{r}}) = -\int_{\infty}^{r} \vec{\mathbf{F}} \circ d\vec{\mathbf{r}}$$

$$F_x = -\frac{dE_p}{dx}$$

$$\vec{\mathbf{F}} = -\frac{\partial E_p}{\partial x}\hat{\mathbf{i}} - \frac{\partial E_p}{\partial y}\hat{\mathbf{j}} - \frac{\partial E_p}{\partial z}\hat{\mathbf{k}} = -\mathbf{grad}\, E_p = -\nabla E_p$$

Operator „nabla”

$$\nabla = \frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}}$$

Problem 1: Potential energy of spring-mass system is given by equation:

$$E_p(r) = \frac{1}{2}kr^2$$

Check, using the following formula:

$$\vec{\mathbf{F}} = -\mathbf{grad}\, E_p$$

if the force of interaction can be expressed as:

$$\vec{\mathbf{F}}(r) = -k\vec{\mathbf{r}}$$

harmonic force

Solution:

$$E_p(r) = \frac{1}{2}kr^2 = \frac{1}{2}k(x^2 + y^2 + z^2)$$

Coordinates of gradient operator:

$$\frac{\partial}{\partial x}E_p(x,y,z) = \frac{\partial}{\partial x}\left(\frac{1}{2}k\left(x^2 + y^2 + z^2\right)\right) = kx$$

$$\frac{\partial}{\partial y}E_p = \frac{\partial}{\partial y}\left(\frac{1}{2}k\left(x^2 + y^2 + z^2\right)\right) = ky$$

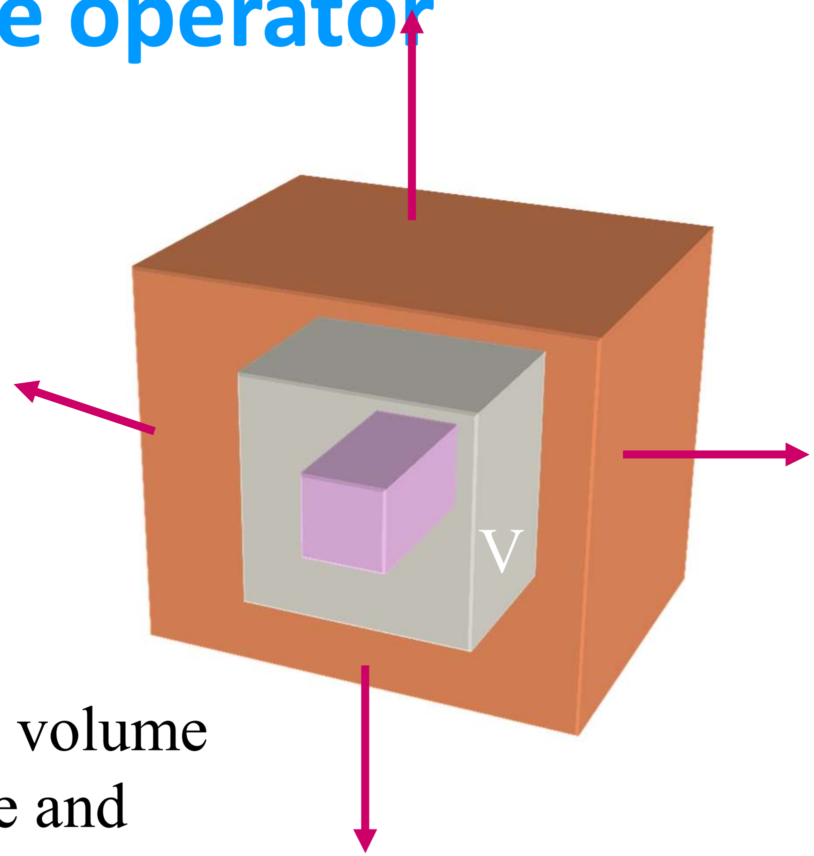
$$\frac{\partial}{\partial z}E_p = \frac{\partial}{\partial z}\left(\frac{1}{2}k\left(x^2 + y^2 + z^2\right)\right) = kz$$

$$\mathbf{grad}\,E_p = kx\,\hat{\mathbf{i}} + ky\,\hat{\mathbf{j}} + kz\,\hat{\mathbf{k}}$$

thus:

$$\vec{\mathbf{F}} = -\mathbf{grad}\,E_p = -k(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}) = -k\vec{\mathbf{r}}$$

Flux and divergence operator



Definition of divergence

$$\operatorname{div}\ \vec{\mathbf{E}} = \lim_{V \to 0} \frac{\oint \vec{\mathbf{E}} \circ d\,\vec{\mathbf{A}}}{V}$$

$\operatorname{div}\vec{\mathbf{E}}$ is, in the limit of infinitely small volume V, flux emerging from the source and determines its efficiency

Gauss-Ostrogradsky theorem

$$\oint_S \vec{\mathbf{E}} \circ d\vec{\mathbf{A}} = \oiiint_V \operatorname{div}\vec{\mathbf{E}}\, dV$$

GAUSS LAW IN DIFFERENTIAL FORM

From Gauss-Ostrogradsky theorem:

$$\oint_S \vec{\mathbf{E}} \circ d\,\vec{\mathbf{A}} = \oiiint_V \operatorname{div} \vec{\mathbf{E}}\, dV$$

Gass law in the integral form :

$$\oint_S \vec{\mathbf{E}} \circ d\,\vec{\mathbf{A}} = \frac{Q_{wew}}{\varepsilon_o} = \frac{1}{\varepsilon_o} \oiiint_V \rho dV$$

Comparing functions under integrals:

charge density

$$\boxed{\operatorname{div} \vec{\mathbf{E}} = \frac{\rho}{\varepsilon_o}}$$

Divergence operator

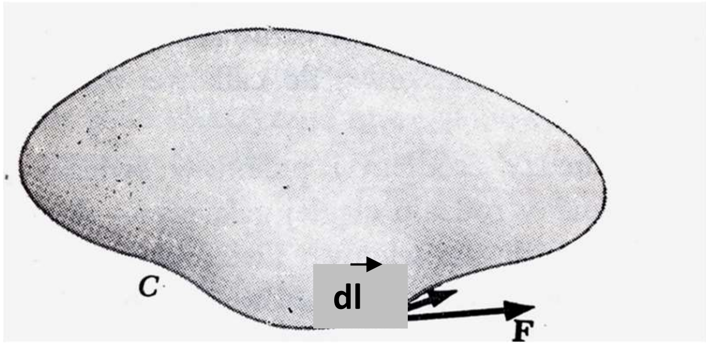
- Divergence of a field vector (w_x, w_y, w_z) in Cartesian coordinate system can be expressed as:

$$div\,\vec{\mathbf{w}} = \nabla \circ \vec{\mathbf{w}} = \frac{\partial w_x}{\partial x} + \frac{\partial w_y}{\partial y} + \frac{\partial w_z}{\partial z}$$

Problem 2: Calculate

$$div\,\vec{\mathbf{r}}$$

CIRCULATION OF VECTOR FIELD



Circulation of vector field $\vec{\mathbf{F}}$ around a closed loop is defined as a line integral:

$$\Gamma = \oint_C \vec{\mathbf{F}} \circ d\vec{\mathbf{l}}$$

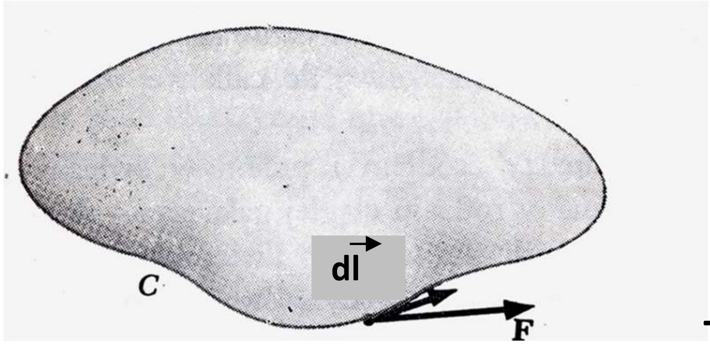
$d\vec{\mathbf{l}}$ the element of integration path has a direction of tangent to the curve C at a given point

If $\vec{\mathbf{F}}$ represents a force, then circulation Γ has a physical sense of work.

If $\vec{\mathbf{F}}$ is a conservative force (electrostatic or gravitational field), then Γ=0.

Curve C encloses a certain surface, bounded by this curve.

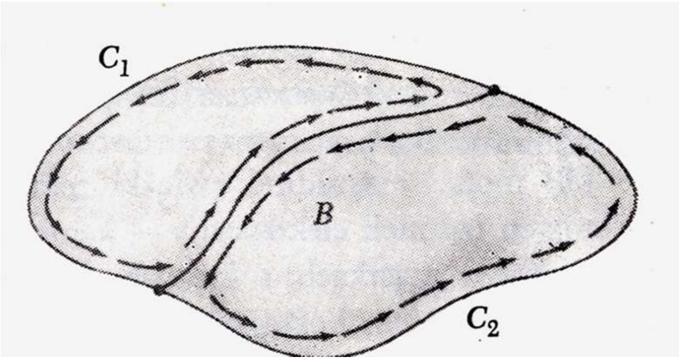
CURL OF THE FIELD



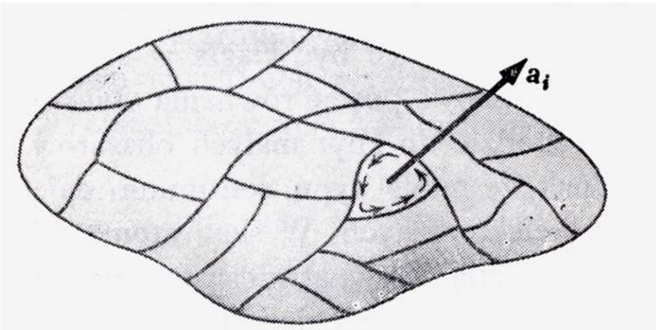
$$\Gamma = \oint_{C} \vec{\mathbf{F}} \circ \mathrm{d}\,\vec{\mathbf{l}}$$

Tracing curve B we create two closed loops

C_1 and C_2 so as :



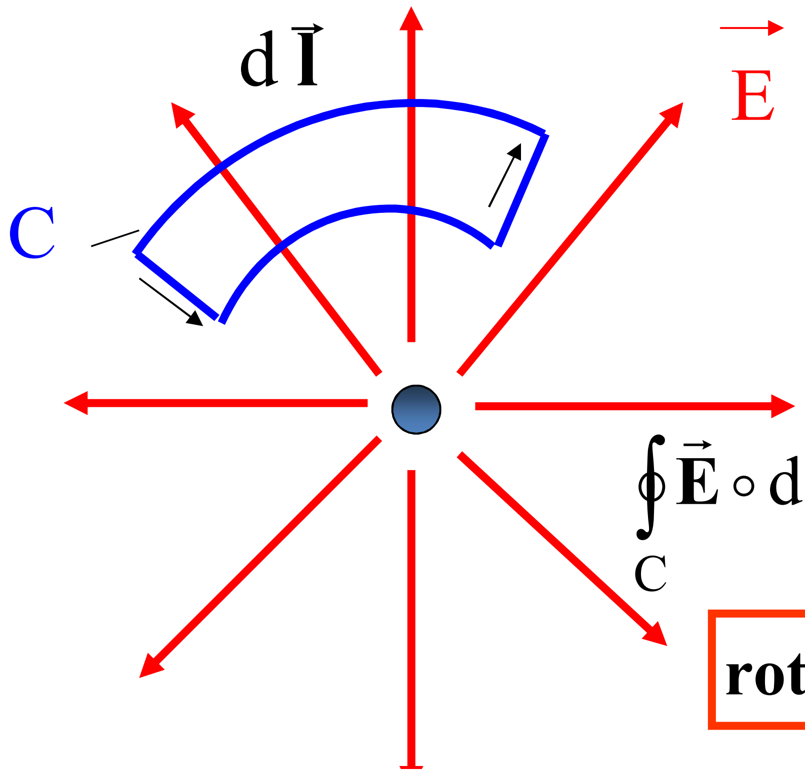
$$\oint_{C} \vec{\mathbf{F}} \circ \mathrm{d}\,\vec{\mathbf{l}} = \oint_{C_1} \vec{\mathbf{F}} \circ \mathrm{d}\,\vec{\mathbf{l}} + \oint_{C_2} \vec{\mathbf{F}} \circ \mathrm{d}\,\vec{\mathbf{l}}$$



<u>Definition of curl operator</u>

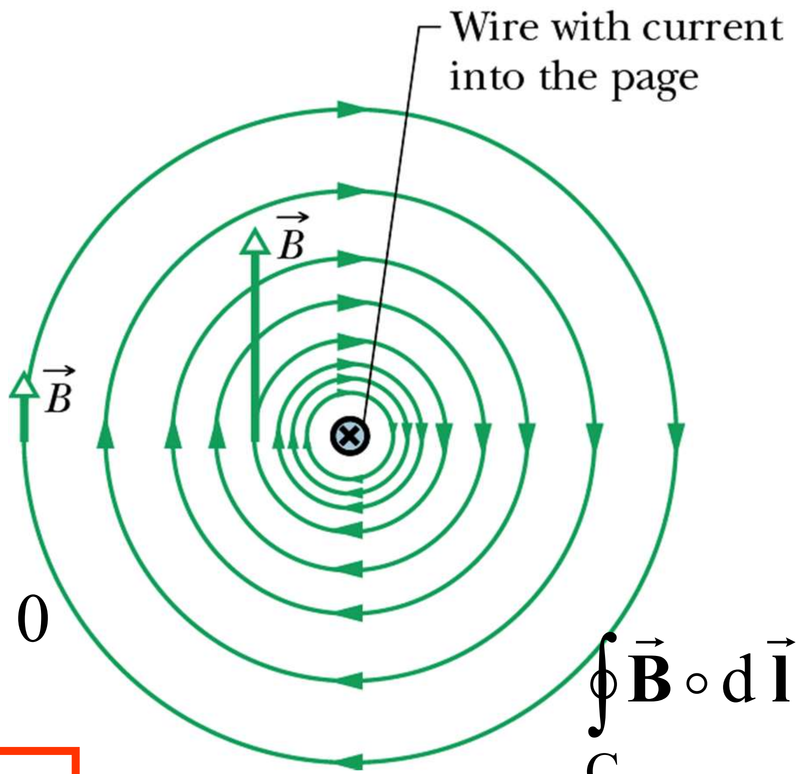
$$(\mathbf{rot}\ \vec{\mathbf{F}}) \circ \hat{\mathbf{n}} = \lim_{a_i \to 0} \frac{\oint_{C_i} \vec{\mathbf{F}} \circ \mathrm{d}\,\vec{\mathbf{l}}}{a_i}$$

Question: Electrostatic field is ***irrotational*** (field rotation is nonzero at each point). What about a rotation of a magnetic field?



$$\oint_C \vec{\mathbf{E}} \circ d\vec{\mathbf{l}} = 0$$

$$\boxed{\mathbf{rot}\,\vec{\mathbf{E}} = 0}$$

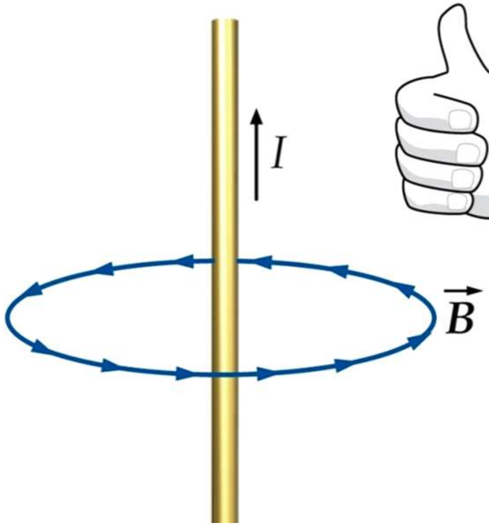


$$\oint_C \vec{\mathbf{B}} \circ d\vec{\mathbf{l}} \neq 0$$

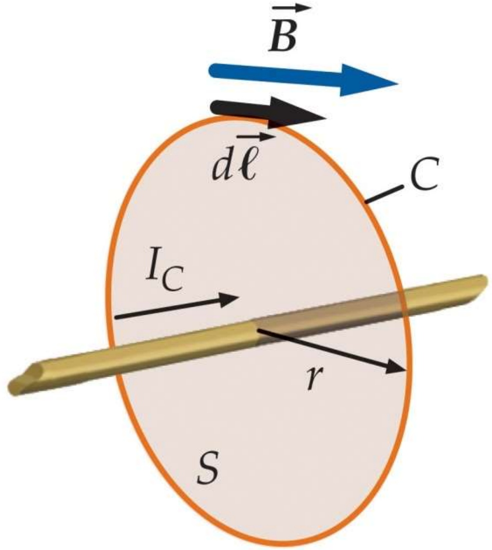
$$\boxed{\mathbf{rot}\,\vec{\mathbf{B}} \neq 0}$$

Answer: In fact, magnetic field is not irrotational. Ampère law

Ampère law



$$\underbrace{\oint_C \vec{\mathbf{B}} \circ d\vec{\mathbf{l}}}_{\text{circulation of magnetic field}} = \mu_o I_C$$

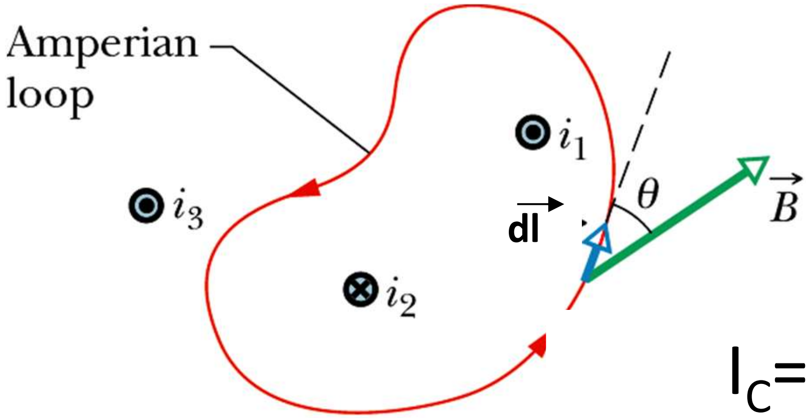


circulation of magnetic field

current inside integration path C

μ_o – magnetic permittivity of vacuum, universal constant

$$\mu_o = 4\pi \cdot 10^{-7}\,\mathrm{T \cdot m / A}$$



$I_C = i_1 - i_2$

Stokes theorem

- Relates circulation of a vector around a curve C to curl at a given point, in a similar way as Gauss-Ostrogradsky theorem relates a flux emanating from a surface to a divergence at a point

$$\oint_C \vec{\mathbf{F}} \circ d\vec{\mathbf{l}} = \underbrace{\oiint_S (\mathbf{rot}\,\vec{\mathbf{F}}) \circ d\vec{\mathbf{a}}}_{\textbf{Surface integral, S is a surface bounded by the curve C}}$$

- Differential form of Ampère law

$$\mathbf{rot}\,\vec{\mathbf{B}} = \mu_o\,\vec{\mathbf{j}}$$

current density

MAXWELL EQUATIONS

Form	Integral	Differential
Gauss law for electricity	$\oint_S \vec{E} \circ d\vec{\mathbf{A}} = \frac{q}{\varepsilon_0}$	$\mathrm{div}\,\vec{\mathbf{E}} = \frac{\rho}{\varepsilon_o}$
Gauss law for magnetism	$\oint_S \vec{\mathbf{B}} \circ d\vec{\mathbf{A}} = 0$	$\mathrm{div}\,\vec{\mathbf{B}} = 0$
Ampere-Maxwell law	$\oint_C \vec{\mathbf{B}} \circ d\vec{\mathbf{l}} = \mu_o(i + \varepsilon_o \frac{d\Phi_E}{dt})$	$\mathrm{rot}\,\vec{\mathbf{B}} = \mu_o(\vec{\mathbf{j}} + \varepsilon_o \frac{\partial \vec{\mathbf{E}}}{\partial t})$
Faraday law	$\oint_C \vec{\mathbf{E}} \circ d\vec{\mathbf{l}} = -\frac{d\Phi_B}{dt}$	$\mathrm{rot}\,\vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$

ELECTROMAGNETIC WAVE IN VACUUM

- We assume that j=0, ρ=0
- Maxwell equations take a form:

$$\mathrm{div}\ \vec{\mathbf{E}} = 0 \quad \Longrightarrow \quad \nabla \circ \vec{\mathbf{E}} = 0$$

$$\mathrm{div}\ \vec{\mathbf{B}} = 0 \quad \Longrightarrow \quad \nabla \circ \vec{\mathbf{B}} = 0$$

$$\mathrm{rot}\ \vec{\mathbf{B}} = \mu_o\, \varepsilon_o \frac{\partial \vec{\mathbf{E}}}{\partial t} \quad \Longrightarrow \quad \nabla \times \vec{\mathbf{B}} = \mu_o \varepsilon_o \frac{\partial \vec{\mathbf{E}}}{\partial t}$$

$$\mathrm{rot}\ \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t} \quad \Longrightarrow \quad \nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$$

Derivation of electromagnetic wave equation

$$\nabla \times (\nabla \times \vec{\mathbf{E}}) = -\nabla \times \frac{\partial \vec{\mathbf{B}}}{\partial t} = -\frac{\partial}{\partial t} \nabla \times \vec{\mathbf{B}}$$

but $\nabla \times \vec{\mathbf{B}} = \mu_o \varepsilon_o \frac{\partial \vec{\mathbf{E}}}{\partial t}$ **hence:**

$$\nabla \times (\nabla \times \vec{\mathbf{E}}) = -\mu_o \varepsilon_o \frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2} \qquad (1)$$

Using mathematical identity: $\vec{\mathbf{a}} \times (\vec{\mathbf{b}} \times \vec{\mathbf{c}}) = (\vec{\mathbf{a}} \circ \vec{\mathbf{c}})\vec{\mathbf{b}} - (\vec{\mathbf{a}} \circ \vec{\mathbf{b}})\vec{\mathbf{c}}$

$$\nabla \times (\nabla \times \vec{E}) = (\underbrace{\nabla \circ \vec{E}}_{0})\nabla - \nabla^2 \vec{E} \qquad (2)$$

Combining (1) and (2) we obtain:

$$\boxed{\nabla^2 \vec{\mathbf{E}} = \mu_o \varepsilon_o \frac{\partial^2 \vec{\mathbf{E}}}{\partial t^2}}$$

General wave equation:

$$\nabla^2 \Psi(\vec{\mathbf{r}}, t) = \frac{1}{v^2} \frac{\partial^2 \Psi}{\partial t^2}$$

VELOCITY OF ELECTROMAGNETIC WAVE IN VACUUM

For magnetic field

$$\nabla^2 \vec{\mathbf{B}} = \mu_o \varepsilon_o \frac{\partial^2 \vec{\mathbf{B}}}{\partial t^2}$$

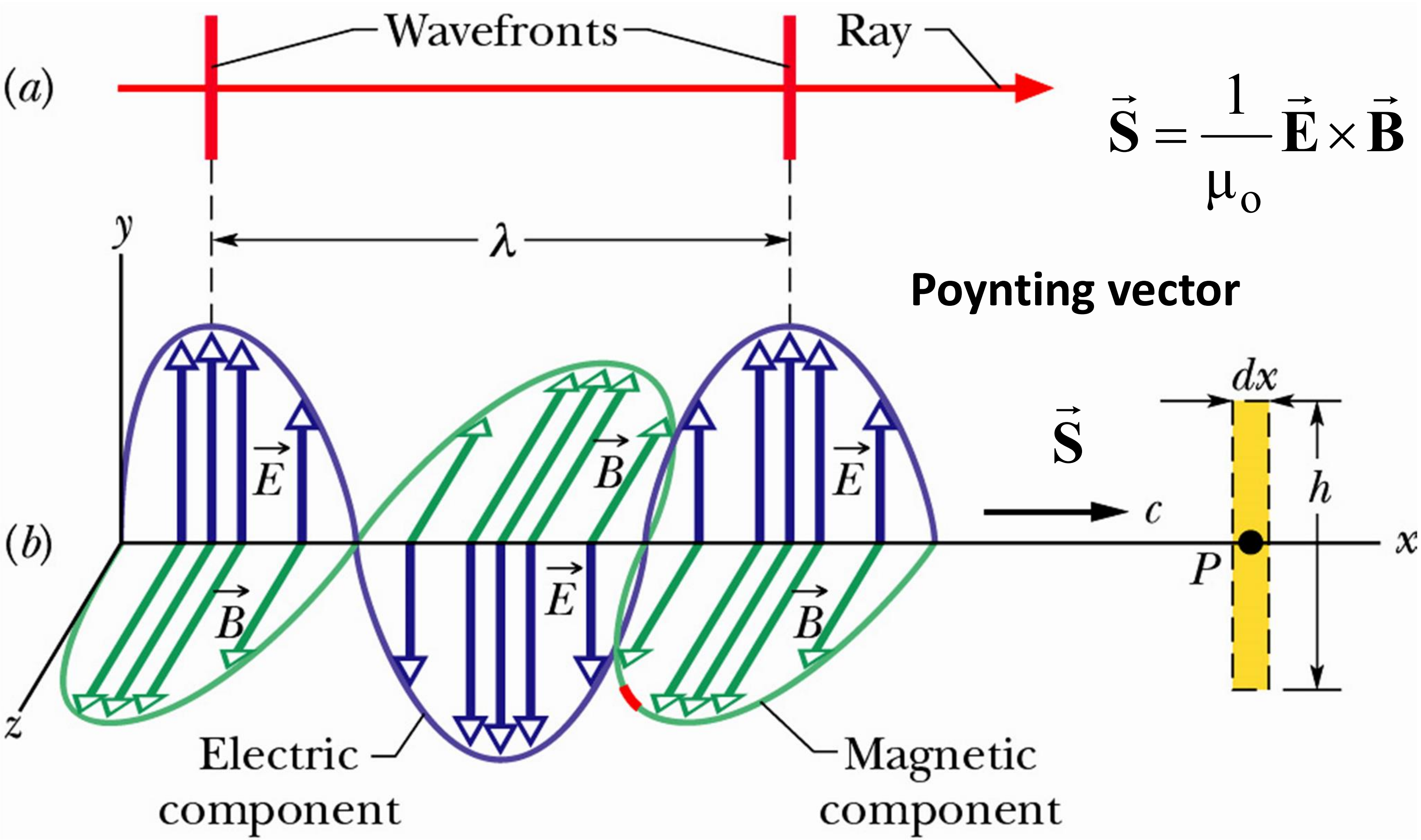
Along with $\nabla^2 \vec{\mathbf{E}} = \mu_o \varepsilon_o \dfrac{\partial^2 \vec{\mathbf{E}}}{\partial t^2}$ constitute electromagnetic wave equations

Perturbation ψ is represented by electric field vector **E** or magnetic induction **B** and the phase velocity v is determined by universal constants, only:

$$\boxed{v = \frac{1}{\sqrt{\mu_o \varepsilon_o}} = c}$$

Velocity of EB wave (velocity of light) in vacuum can be predicted theoretically as $c \approx 3 \cdot 10^8$ m/s

Propagation of EB wave



$$\vec{\mathbf{S}} = \frac{1}{\mu_o}\vec{\mathbf{E}} \times \vec{\mathbf{B}}$$

Poynting vector